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MULTIFRACTAL ANALYSIS OF CHAOTIC POINT SETS

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It is generally acknowledged that non-linear systems exhibit important features that cannot be understood, controlled, or exploited without addressing their behavior in a non-linear form. We describe qualitative and quantitative techniques for the description of non-linear dynamic systems. These techniques are illustrated through the analysis of symmetric chaos.

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INTRODUCTION

Many important physical systems[1] exhibit highly unstable or chaotic behavior. Traditional methods for treating such systems have been linearization and stability analysis. It is generally acknowledged that non-linear systems exhibit important features that cannot be understood, controlled, or exploited without addressing their behavior in a non-linear form. Thus, techniques are being developed to model and analyze such complex systems. The approach entails the analysis of complicated structures that exhibit fractal geometry. Fractal parameters describe how the length, area, and/or density of an object varies with scale and may provide quantitative measures of the complex structures that occur in nature. To optimally exploit fractal perspectives, quantitative techniques for measurement of fractal properties are evolving. In this report, qualitative and quantitative techniques are applied to elucidate the fractal structure of the strange attractors that govern the evolution of symmetric chaos in some model systems.

BACKGROUND

The qualitative features of the time evolution of many physical systems are embodied in the solutions to equations of the form:

$$\dot{x}(t) = F(x(t), a, b, \ldots) \qquad (1a)$$

where a, b, etc. are parameters and $x \in R^n$. The function F determines, in principle, the non-linear evolution of the system. In practice, the behavior of non-linear dynamical systems is quite complex, and the solutions to Eq. (1a) (i.e., the trajectories x(t)) are frequently extremely sensitive to small changes in initial conditions or the system parameters. Equation (1a) is a continuous time description of the dynamical system, however, such systems are generally treated in terms of difference equations or discrete time mappings of the form

$$x(n+1) = F(x(n), x(n-1), \ldots, a, b, \ldots) \qquad (1b)$$

The motion of non-linear systems, governed by Eq. (1b), may be unbounded or may settle down after an initial "transient" to a subset of R^n called an attractor. The motion of such systems can be characterized by their attractors rather than by specific trajectories. The attractors of non-linear dynamical systems fall into two categories:

i. Periodic attractors. When an attractor comprises a finite set of points (that are visited in sequence), the motion of the system is periodic and one refers to a periodic attractor. If the number of points is N, it is called a period N attractor.

ii. Chaotic attractors. When an attractor contains an infinite set of points, the motion of the system is called "chaos," trajectories are called "chaotic," and one refers to "strange attractors" or "chaotic attractors." Strange attractors have fractal structure.

The attractors studied here describe the dynamics of an important subset of chaotic systems: symmetric chaotic mappings in E^2, which were qualitatively characterized by Field and Golubitsky[2]. In particular, we focus our efforts on the quantitative determination of the fractal parameters that characterize the attractors of mappings, which transform from D_n to Z_n symmetry as a parameter is changed. Field and Golubitsky[2] referred to the motion of non-linear systems, whose attractors were invariant with respect to such symmetry groups, as "symmetric chaos."

Box-counting algorithms for fractal analysis are based on a statistical description of the variation with scale of the distribution of "mass" of a fractal set. We employ a box-counting technique to extract the fractal parameters that characterize the attractors of four systems exhibiting symmetric chaos.

APPROACH

The attractors of the polynomial mappings determined by iteration of

$$z \rightarrow (\lambda + \alpha\,|z|^2 + B\,Re(z^n) + i\,\omega)\,z + \gamma\,(z^*)^{n-1} \qquad (2)$$

where $z \in \acute{C}$, and λ, α, B, γ, and n are real constants, have D_n symmetry for $\omega=0$, and Z_n symmetry for $|\omega| > 0$. Field and Golubitsky[2] recently discussed qualitative techniques for analyzing the attractors of polynomial mappings.

The nature of the attractors of Eq. (2) is elucidated by generating graphical representations of their structure, similar to those of Ref. 2, and by fractal analysis.

The purpose of the graphical representations is to visually illustrate the chaotic structure and symmetry of the attractors studied. Although a number of subtle decisions must be made in such representations, only standard techniques are employed.

The fractal analysis provides quantitative measures of the attractor structures. The fractal analysis employs the agglomeration box-counting (ABC) algorithm of Meisel, Johnson, and Cote[3]. ABC is especially suitable for analysis of large point sets.

Since the ABC algorithm is not a standard procedure, we outline the technique here. ABC employs a set of hypercubes (boxes) of linear dimension E to cover a subset of the fractal set. For each E, the partition function

$$Z(q;E) = \sum_i p_i^{\,q}(E) \qquad (3)$$

is computed for a range of the descriptive index q, and $p_i(E)$ is the occupation probability for the i^{th} box of side E. The partition function is related to the Hentschel and Procaccia[4,5] fractal dimension D(q) by the scaling relationship:

$$Z(q;E) \sim E^{(q-1)\,D(q)} \qquad (4)$$

where D(q) is a generalization of the Mandelbrot fractal dimension[6]. At q=0, D(q) reduces to the standard Mandelbrot fractal dimension. If D(q) is non-integer and invariant, the set is monofractal. If D(q) varies with q, it is said to be multifractal.

The essential steps in applying ABC are:

i. Define a set of elementary boxes for the given point set S, such that the elementary boxes have edge lengths,

$$E(H,K,M,...)=E_0/(2^H\,3^K\,5^M...)$$

where H, K, M, ... are integers, and E_0 is to be the largest box edge length in the set.

ii. Compute (measure, etc.) and store the occupation numbers (integers) $n_i(E(H,K,M,...))$ for the elementary boxes. (The probabilities $p_i(E)=n_i(E)/N$ need not be stored.)

iii. For each $h \in \{0,1,...,H\}$, $k \in \{0,1,...,K\}$, $m \in \{0,1,...,M\}$, ... define sets of boxes of edge length

$$E(h,k,m,...) = E_0/(2^h\, 3^k\, 5^m ...),$$

which also cover the point set and contain integer numbers of elementary boxes.

a. Compute the occupation numbers $n_i(E(h,k,m,...))$ for each box by summing the occupation numbers in the elementary boxes contained.

b. For a representative set of q values, compute $Z(q,E(h,k,m,...))$ from Eq. (3).

iv. For each q value, compute D(q) by least squares fitting of a straight line to $\ln(Z(q,E(h,k,m,...))/(q-1)$ vs $\ln(E(h,k,m,...))$ as follows from Eq. (4). In applications, the elementary box size must be large enough to contain a "substantial number" of members of the set, and the largest boxes included in the fit must be smaller than the extent of the point set. (Special techniques are required for $q=1$, but D(q) is a smooth function of q in any case.)

ABC has been tested on Euclidean point sets and simple monofractal and multifractal constructs for which analytic values of D(q) could be computed. Worst case results were within 5 percent of the analytic values for $q \geq 0$ and $N=10^7$ for the cases studied in Ref. 3. Reliable values of D(q) were not obtained employing ABC (or conventional box-counting[4]) for $q<0$; therefore, D(q) for $q<0$ are not reported here.

RESULTS

Results are presented for four classes of symmetric chaotic mappings based on Eq. (2): Figures 1 and 2 pertain to attractors having 3-fold symmetry (i.e., D_3 and Z_3 symmetry), Fig. 3 to attractors having 6-fold symmetry, and Fig. 4 to attractors having 16-fold symmetry.

For each class of mappings defined by a specific choice of λ, α, B, γ, and n, attractors were generated based on 10^7 iterations of Eq. (2) for ω values starting from 0 and stepping by 0.01 over the range for which the motion is bounded. Smaller changes of ω were taken in "interesting" ranges. The parameter set defining each class of mappings is given in the captions of the a-parts of each figure. (The λ, α, B, γ, and n parameter sets were chosen to match the Field and Golubitsky[2] D_n mappings.)

Two-dimensional images of subsets of the attractors (generated for $\omega=0$ and selected other values of ω), which were formed using the density of the attractor mapping as an index to a gray scale using a technique similar to Field and Golubitsky[2], are shown for each case. The attractors are depicted in square regions whose corners are at (-2, -2) and (2, 2) for the 3-fold mappings and (-1,-1) and (1,1) for the others. Each image is based upon 768 x 768 pixels shaded according to number of hits modulo 512.

Three-dimensional images, in which the smoothed density of the attractor is the third coordinate, are shown for the 6- and 16-fold symmetric attractors to enhance visualization. Although a more faithful representation was obtained prior to smoothing, unsmoothed density is not shown because the rapid (chaotic) density variations were difficult to interpret.

Graphs of the Hentschel-Procaccia[5] generalized fractal dimension D(q) (for q=0, 2, 5, and 9) vs ω over the range for which chaotic attractors exist and D(q) vs q for the ω values corresponding to the chaotic attractor images are shown for all cases.

For all the mappings presented:

i. For $\omega>1$, unbounded motion results.

ii. For $\omega<1$,

a. a range of values of ω for which periodic orbits, possibly comprising parts of period doubling cascades with decreasing ω, was found;

b. chaotic attractors were found for a range of positive ω extending from 0;

c. dramatic changes in the structure of the attractors with varying ω were apparent in their two- and three-dimensional images.

D(0) is essentially invariant under changes in ω except for the mappings of Fig. 1; however, D(q) exhibits substantial non-monotonic variation with ω over the chaotic range for all classes of mappings studied.

The 3-fold symmetric mappings of Figs. 1 and 2 are most interesting:

The mappings of Fig. 1 yield a single basin of attraction at small ω that fissions into three symmetrically disposed sub-basins of attraction for ω near 0.33. Near the "critical point" one passes through a "periodic window." An enhanced rendering (i.e., each point is rendered as a small circle) of one of the three period 8 sub-attractors in the periodic window at ω=0.338500... is shown. An image of a corresponding chaotic sub-attractor at ω=0.39 is also shown. The full attractors in the three sub-basin range have Z_3 symmetry. However, the motion of the dynamical system would be confined to one of the sub-basins of attraction and would not have Z_3 symmetry. Searches based on changes in ω of 0.001 did not yield periodic orbits with a period other than 8 ($=2^3$) in the window. (Although the ABC algorithm yielded values as large as 0.06, D(q) values are shown as 0.00 for ω in the strictly periodic windows in Figs. 1 and 2.)

The mappings of Fig. 2 yield a single basin of attraction or unbounded orbits for almost all ω. Two strictly periodic windows were found, and their attractors are shown. The window at ω=0.06 contains a period 2 attractor in the midst of a one basin chaotic range of ω. The window at ω=0.12 contains a period 39 (=3*13) attractor, which has Z_3 symmetry, but the factor of 13 is puzzling. The window at ω=0.28 contains a quasi-period 12 attractor, which has Z_3 symmetry, but is not strictly periodic even after 10^6 iterations. The ABC algorithm yields values of D(q) greater than 0.30 in accord with the apparent chaotic nature of the attractor for ω=0.28.

The mappings of Figs. 3 and 4 yield substantial variations in structure and multifractal D(q) with ω. Although prominent dips were apparent in D(q) near specific values of ω, no periodic windows were discovered searching via steps of 0.001 in ω near those dips.

CONCLUSIONS

The results presented in Figs. 1 through 4 comprise a qualitative and quantitative solution (except for the possible omission of some narrow periodic windows) of the non-linear problems defined. One needs both approaches to characterize the dynamics.

Even the simplest non-linear dynamical systems exhibit unexpectedly complex behavior. Indeed, even in retrospect, it seems impossible to deduce the complex form of the symmetric chaotic attractors or the diversity of D(q) vs q for representative ω-values, such as those illustrated in Figs. 1 through 4, by analysis of the form of the mappings of Eq. (2).

The nature of the change in symmetry from D_n to Z_n is generally not reflected in the Mandelbrot fractal dimension D(0), but it is manifested in changes of the Hentschel and Procaccia[3,4,5] generalized fractal dimension D(q). Thus, both qualitative and multifractal characterization of the attractors of non-linear systems of interest are recommended.

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6. For example, see B.B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, New York (1982).

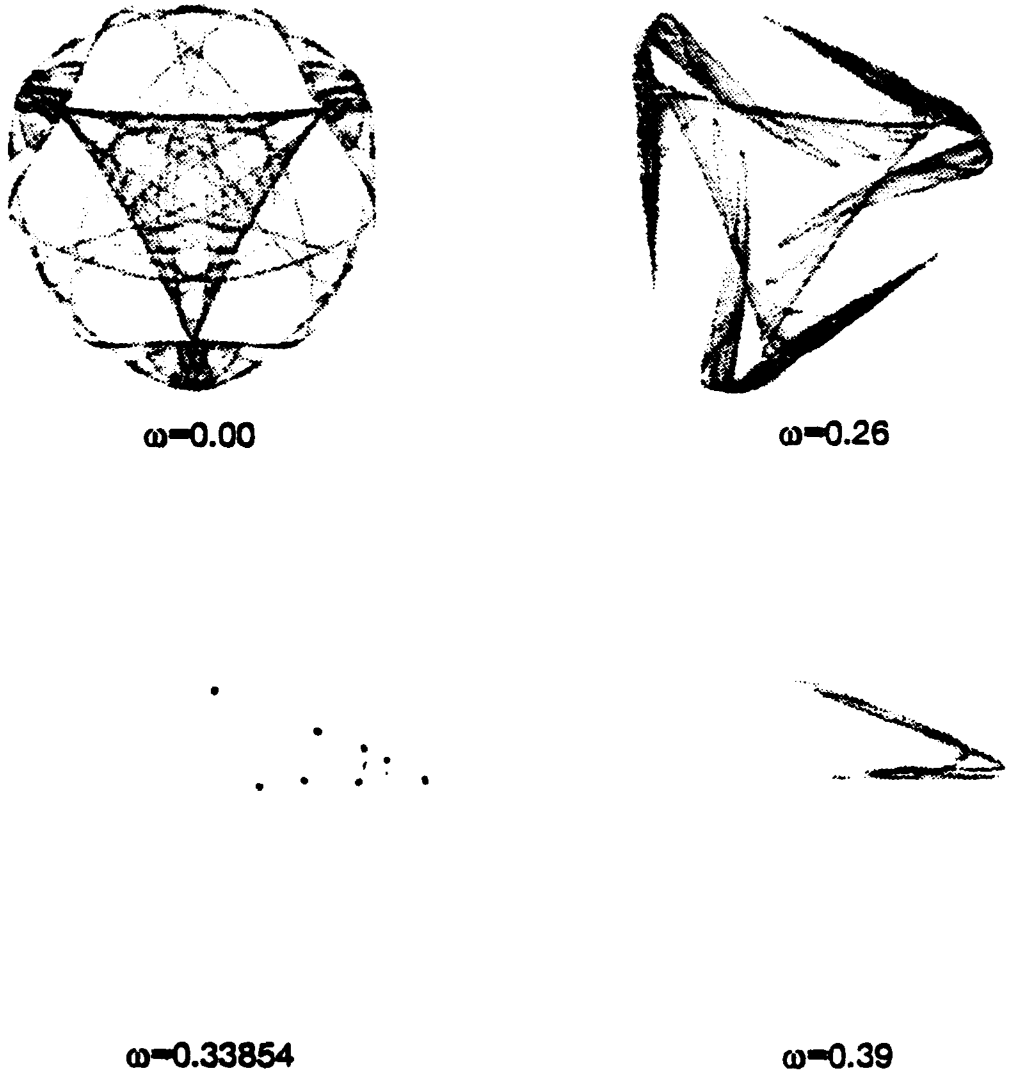

Figure 1a. Attractor with $n=3$; $\lambda=1.56$; $\alpha=-1.0$; $B=0.1$; $\gamma=-0.82$.

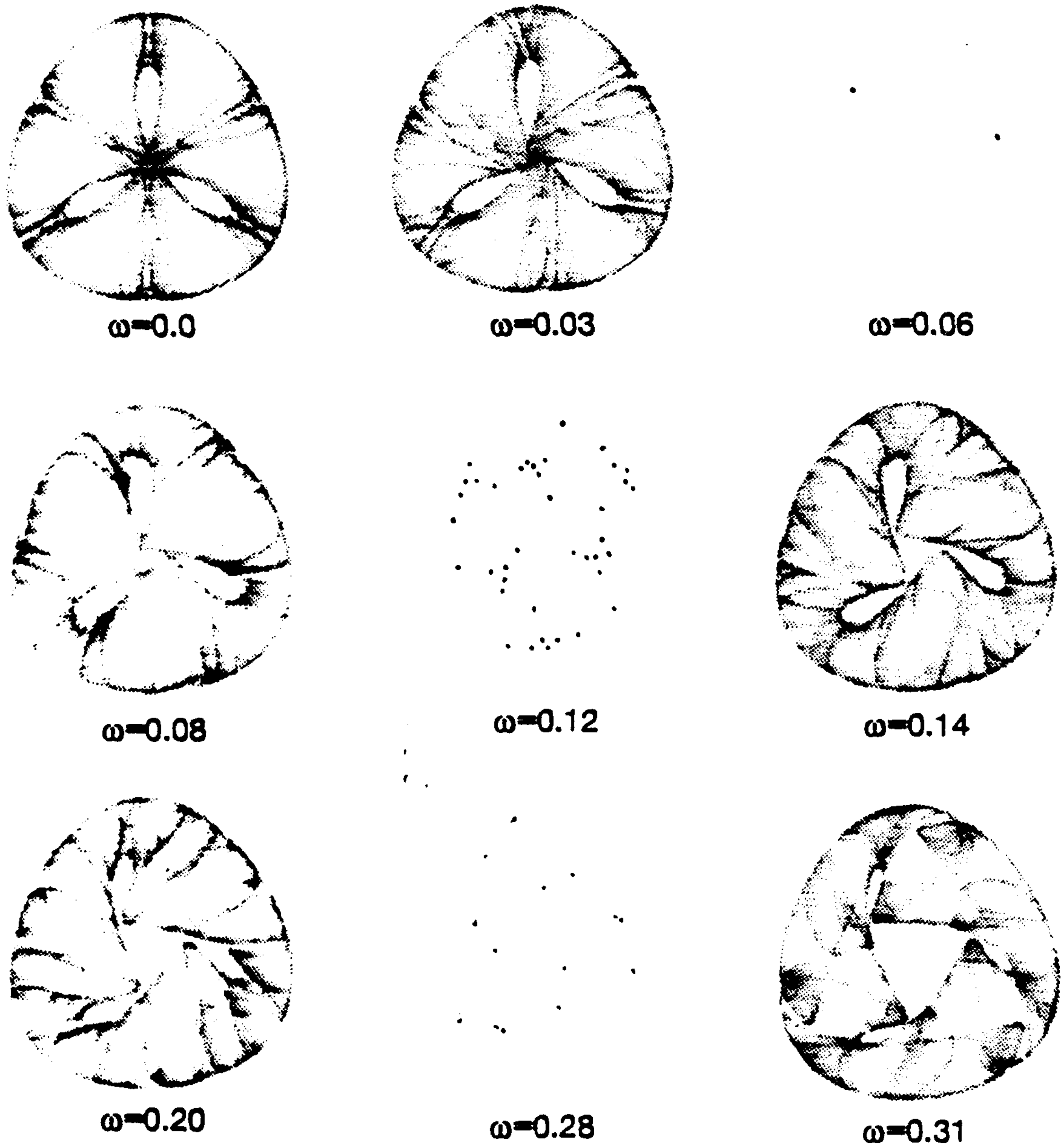

Figure 2a. Attractor with $n=3$; $\lambda=-2.38$; $\alpha=1.0$; $B=0.0$; $\gamma=0.1$.

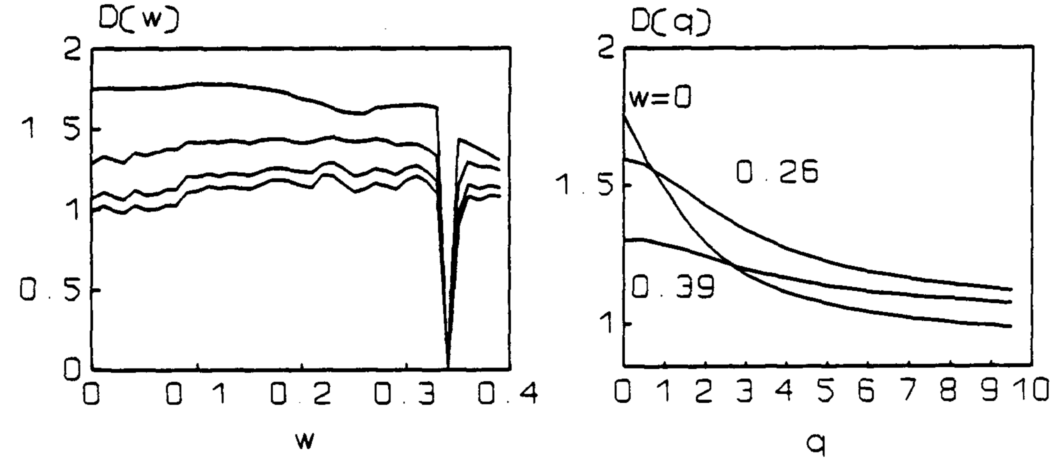

Figure 1b. Fractal analysis of the attractors depicted in Fig. 1a. (D is non-increasing in q and for all attractors D(w) are shown for q={0,2,5,9}.)

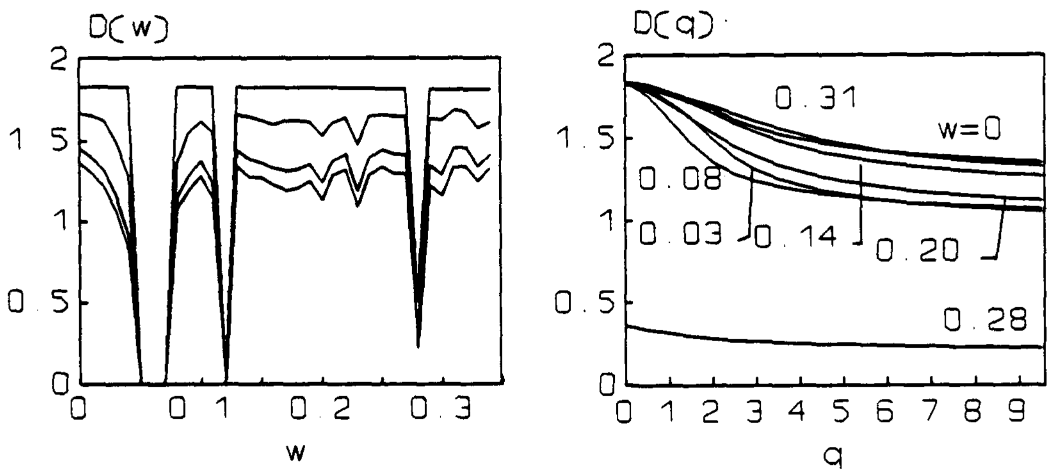

Figure 2b. Fractal analysis of the attractors depicted in Fig. 2a.

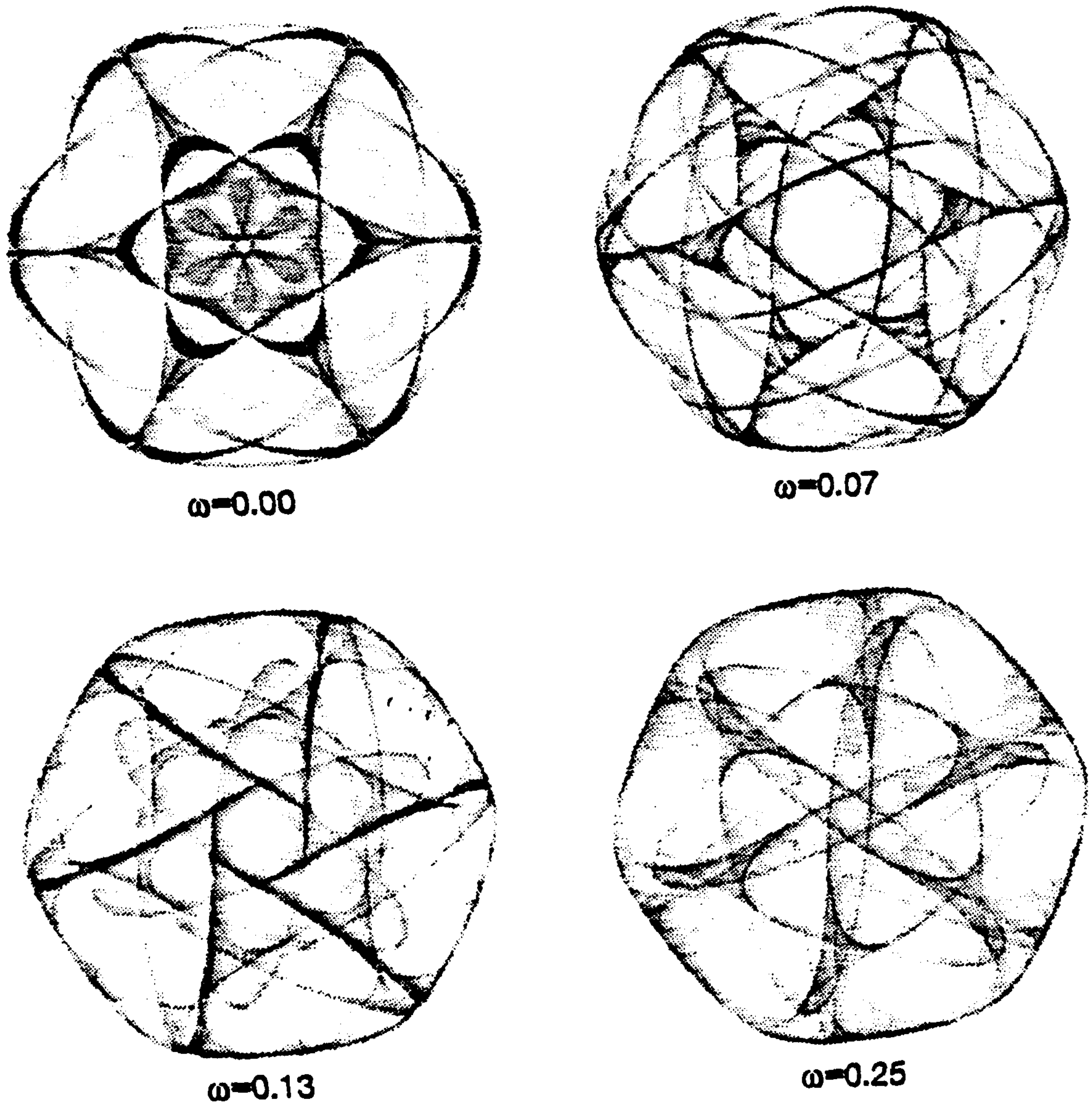

Figure 3a. Attractor with $n=6$; λ-2.7; $\alpha=5.0$; $B=2.0$; $\gamma=1.0$.

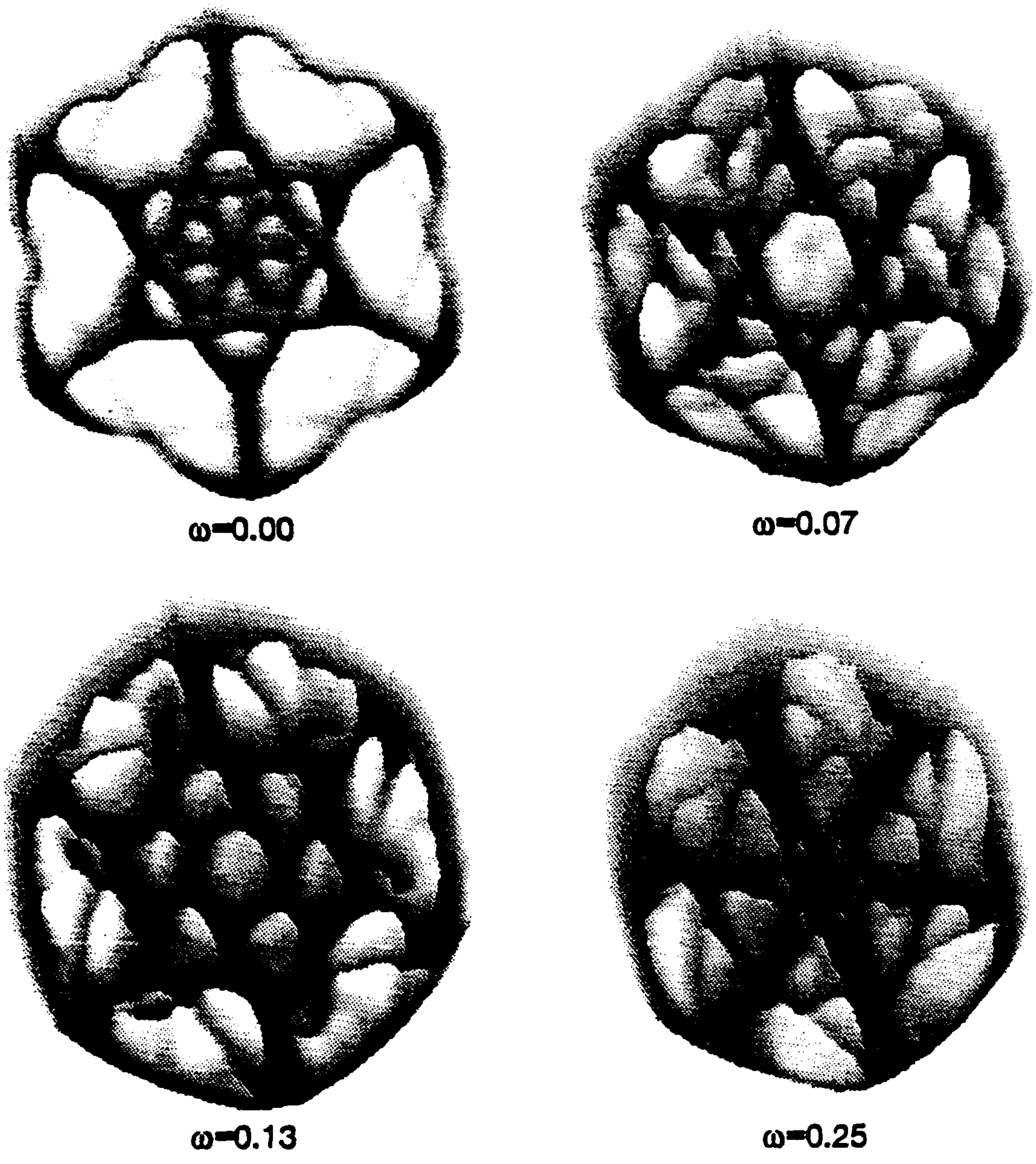

Figure 3b. Three-dimensional representation of attractor in 3a.

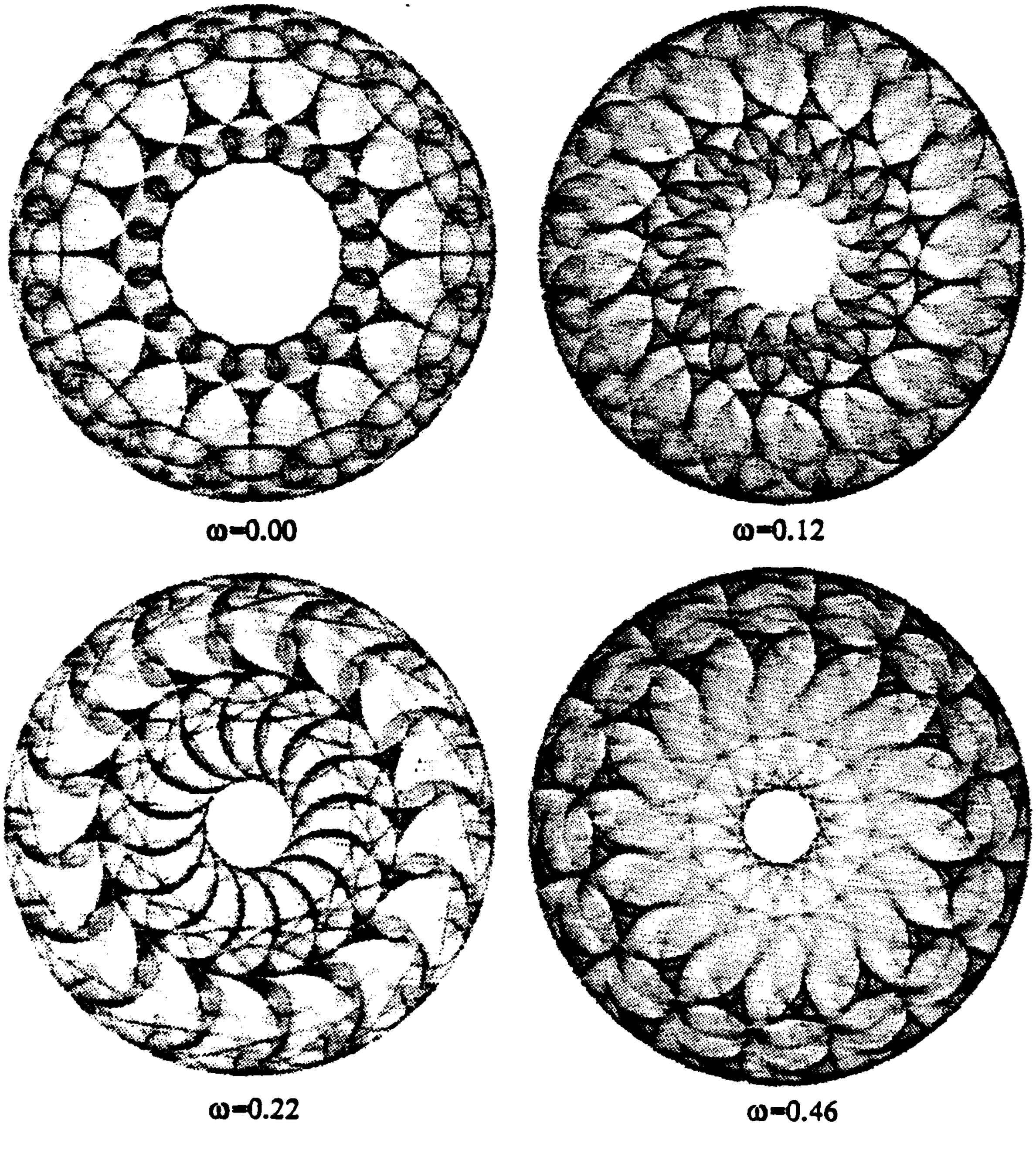

Figure 4a. Attractor with n=16; λ=2.39; α=-2.5; B=-0.1; γ=0.9.

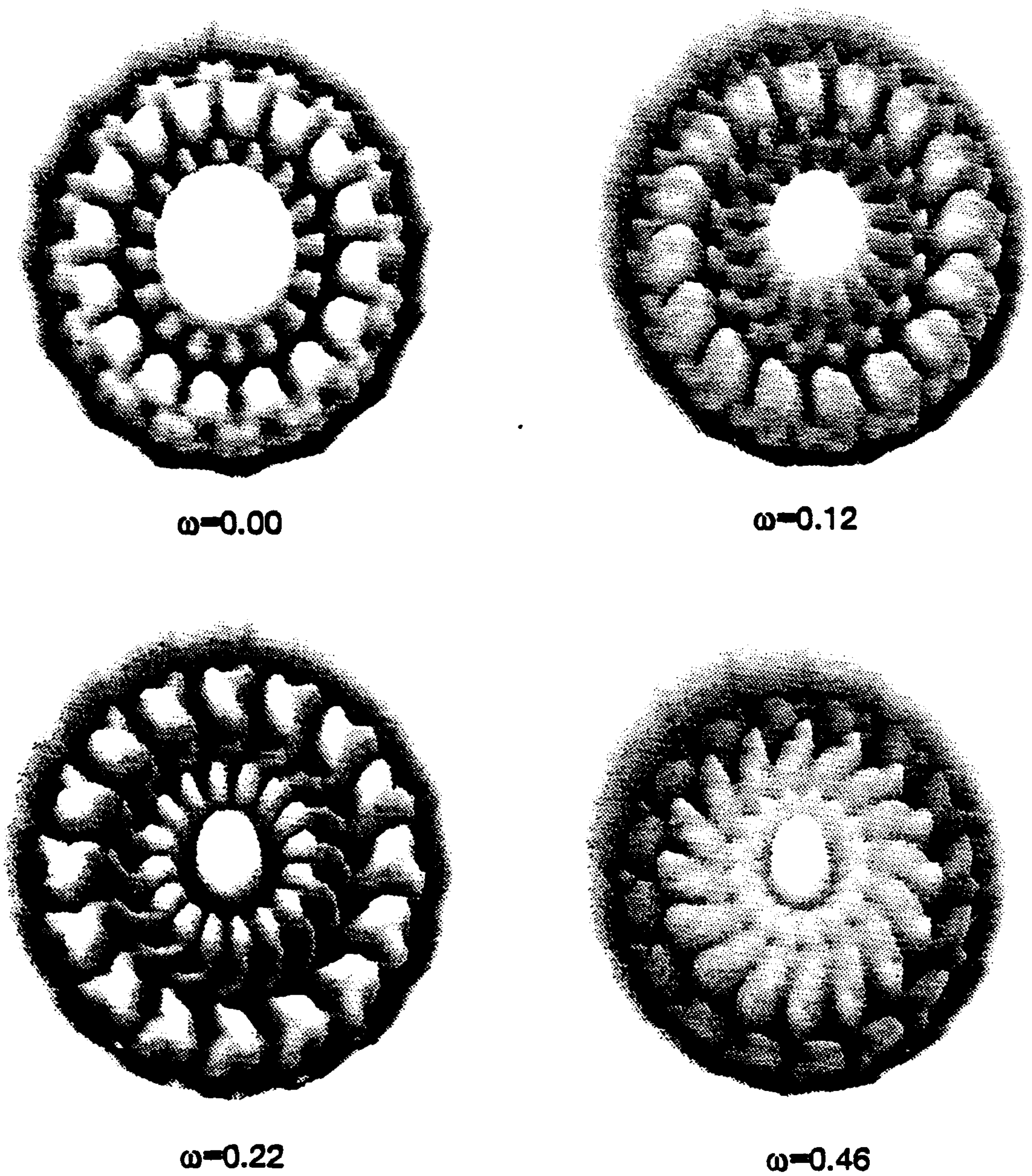

Figure 4b. Three-dimensional representation of attractor in 4a.

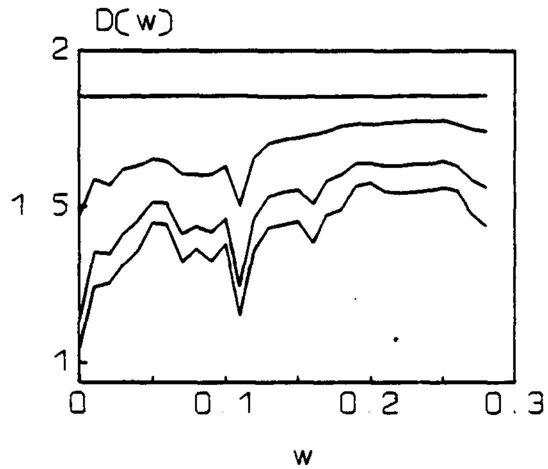

Figure 3c. Fractal analysis of the attractors depicted in Figs. 3a and 3b.

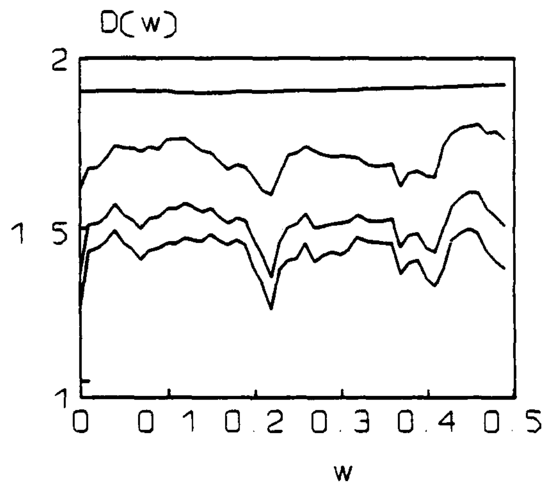

Figure 4c. Fractal analysis of the attractors depicted in Figs. 4a and 4b.

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